Introduction to Holomorphic Floer Theory: brane quantization, exponential integrals and resurgence II

Yan Soibelman

KANSAS STATE UNIVERSITY

RenewQuantum seminar, April 20, 2021

Recall the following isomorphisms and their categorification from last time:
1) (Betti local to global isomorphism). Outside of the Stokes rays (where ≥ 2 points z_i, z_j belong to a ray $Arg(\hbar) = const$) we have an isomorphism of local systems of abelian groups on $\mathbb{C}^*_\hbar$:

$$i_{Betti,\hbar} : H_{Betti,\hbar} \simeq \oplus_{i\in S} H_{Betti,loc,z_i/\hbar}.$$

2) (de Rham local to global isomorphism). We have the natural isomorphism of $\mathbb{C}((\hbar))$-vector spaces:

$$i_{DR,loc} : H_{DR,\hbar} \otimes \mathbb{C}((\hbar)) \simeq \oplus_{i\in S} H_{DR,loc,z_i/\hbar} \otimes \mathbb{C}((\hbar)).$$

3) (Global Betti to de Rham). For each $\hbar \in \mathbb{C}^*$ we have:

$$iso_\hbar : H_{Betti,\hbar} \otimes \mathbb{C} \simeq H_{DR,\hbar}.$$

It gives rise to an isomorphism of holomorphic vector bundles on $\mathbb{C}^*_\hbar$.
4) (Local Betti to de Rham). For each $i \in S$ we have an isomorphism of $\mathbb{C}((\hbar))$-vector spaces:

$$iso_{loc} : H_{Betti,loc,z_i/\hbar} \otimes \mathbb{C}((\hbar)) \simeq H_{DR,loc,z_i/\hbar} \otimes \mathbb{C}((\hbar)).$$

1') (Fukaya local to global). Outside of Stokes rays in $\mathbb{C}^*_\hbar$ we have an isomorphism of analytic families of categories:

$$\mathcal{F}_\hbar \simeq \mathcal{F}_{\hbar,loc}.$$

Here Stokes rays are those rays $Arg(\hbar) = const$ for which there exist pseudo-holomorphic discs with boundaries on L_0 and L_1. One can show that they agree with the Stokes rays in 1).

2') (Holonomic local to global).

$$Hol_\hbar \otimes \mathbb{C}((\hbar)) \simeq Hol_{loc},$$

where in the LHS the notation means that corresponding category over $\mathbb{C}[\hbar]]$ with inverted $\hbar$.

3') (Global Riemann-Hilbert correspondence). We have an equivalence of analytic families of categories over $\mathbb{C}^*_\hbar$:

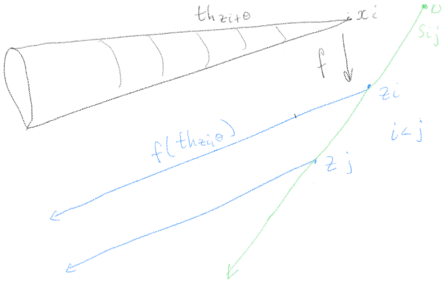
$$Hol_\hbar \simeq \mathcal{F}_\hbar.$$

4') (Local Riemann-Hilbert correspondence).

$$Hol_{loc} \simeq \mathcal{F}_{\hbar,loc} \otimes \mathbb{C}((\hbar)).$$

Thimbles

Assume now that X is Kähler and f is Morse. Let $\theta = Arg(\hbar)$. We define a thimble $th_{z_i,\theta+\pi}$ as the union of gradient lines (for the Kähler metric) of the function $Re(e^{-i\theta}f)$ outcoming from the critical point $x_i \in X$ such that $f(x_i) = z_i$.The same curve is an integral curve for the Hamiltonian function $Im(e^{-i\theta}f)$ with respect to the *symplectic* structure. Hence $f(th_{z_i,\theta+\pi})$ is a ray $Arg(z) = \theta + \pi$ outcoming from the critical value $z_i \in S$.



$th_{z_i+\theta}$
x_i
0
S_{ij}
f
z_i
$i<j$
$f(th_{z_i\theta})$
z_j

Let us assume that X carries a holomorphic volume form *vol* and define the collection of exponential integrals for all $\hbar \in \mathbb{C}^*$ which do not belong to Stokes rays $Arg(\hbar) = Arg(z_i - z_j), i \neq j$:

$$I_i(\hbar) = \int_{th_{z_i,\theta+\pi}} e^{f/\hbar} vol.$$

Assume that the set of critical values $S = \{z_1, ..., z_k\}$ is in generic position in the sense that no straight line contains three points from S. Then a Stokes ray contains two different critical values which can be ordered by their proximity to the vertex.

Wall-crossing formulas

It is easy to see that if in the $\hbar$-plane we cross the Stokes ray $s_{ij} := s_{\theta_{ij}}$ containing critical values $z_i, z_j, i < j$, then the integral $I_i(\hbar)$ changes such as follows:

$$I_i(\hbar) \mapsto I_i(\hbar) + n_{ij} I_j(\hbar),$$

where $n_{ij} \in \mathbb{Z}$ is the number of gradient trajectories of the function $Re(e^{i(Arg(z_i - z_j)/\hbar)} f)$ joining critical points x_i and x_j.

Let us modify the exponential integrals such as follows:

$$I_i^{mod}(\hbar) := \left(\frac{1}{2\pi\hbar}\right)^{n/2} e^{-z_i/\hbar} I_i(\hbar).$$

Then as $\hbar \to 0$ the stationary phase expansion ensures that as a formal series

$$I_i^{mod}(\hbar) = c_{i,0} + c_{i,1}\hbar + ... \in \mathbb{C}[[\hbar]],$$

where $c_{i,0} \neq 0$. The jump of the modified exponential integral across the Stokes ray s_{ij} is given by $\Delta(I_i^{mod}(\hbar)) = n_{ij} I_j^{mod}(\hbar) e^{-(z_i - z_j)/\hbar}$.

RH problem

Therefore the vector $\bar{I}^{mod}(\hbar) = (I_1^{mod}(\hbar), ..., I_k^{mod}(\hbar)), k = |S|$ satisfies the Riemann-Hilbert problem on $\mathbb{C}$ with known jumps across the Stokes rays and known asymptotic expansion as $\hbar \to 0$ (notice that because of our ordering of the points in S, the function $e^{-(z_i - z_j)/\hbar}$ has trivial Taylor expansion as $\hbar \to 0$ along the Stokes ray s_{ij}).

In abstract terms, we consider a Riemann-Hilbert problem for a sequence of $\mathbb{C}^k$-valued functions (here k is the rank of the Betti cohomology, which is under our assumptions is equal to the cardinality $|S| = k$) $\Psi_1(\hbar), ..., \Psi_k(\hbar)$ on $\mathbb{C}^* - \cup(Stokes\, rays)$ each of which has a formal power asymptotic expansion in $\mathbb{C}[[\hbar]]$ as $\hbar \to 0$, and which satisfy the following jumping conditions along the Stokes rays s_{ij}:

$$\Psi_j \mapsto \Psi_j,$$

$$\Psi_i \mapsto \Psi_i + n_{ij} e^{-\frac{z_i - z_j}{\hbar}} \Psi_j.$$

This collection $(\Psi_i)_{1 \le i \le k}$ gives rise to a holomorphic vector bundle.

The above considerations are a special case of the theory of analytic wall-crossing structures discussed in our recent paper with Maxim arXiv: 2005.10651. General resurgence conjecture formulated there explains in particular the resurgence of formal expansions of exponential integrals. From the point of view of that conjecture the resurgence is equivalent to gluing of the holomorphic vector bundle over $\mathbb{C}_\hbar$ as explained above. From the point of view of the HFT the resurgence is encoded in the upper bounds for the virtual numbers of pseudo-holomorphic discs with the boundary on $L_0 \cup L_1$. Such discs appear when $\hbar$ crosses the Stokes rays.

Abstractly it is best of all described in the language of stability data on graded Lie algebras. In the case of resurgent series it is the Lie algebra of vector fields on the "torus of characters of the charge lattice".

On the next three slides I will review this formalism and illustrate it in the example of exponential integrals.

Stability data

The formalism of stability data on graded Lie algebras was proposed for studying algebraic and analytic properties of generating functions arising in Donaldson-Thomas theory, if we understand it in the sense of our original paper with Maxim arXiv:0811.2435.
Here is the list of data and properties:
i) free abelian group of finite rank Γ (charge lattice);
ii) graded Lie algebra $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ over $\mathbb{Q}$;
iii) homomorphism of abelian groups $Z : \Gamma \to \mathbb{C}$ (central charge);
iv) collection of elements $a(\gamma) \in \mathfrak{g}_\gamma, \gamma \in \Gamma - \{0\}$ (rational enumerative invariants).
These data are required to satisfy one axiom called Support Property. Roughly, it says that support of the collection $(a(\gamma))_{\gamma \in \Gamma - \{0\}}$ is sufficiently far from $Ker\, Z$.

For $\mathfrak{g} := \mathit{Vect}_\Gamma$, the graded Lie algebra of vector fields on the torus $\mathbf{T}_\Gamma = \mathit{Hom}(\Gamma, \mathbb{C}^*)$, stability data can be restated in terms of the gluing data of a certain formal scheme (see our arXiv:1303.3253). If it comes from a complex analytic space, the stability data are called analytic. Using the $\mathbb{C}^*$-action $Z \mapsto Z/\hbar$ one can construct an analytic fiber bundle over $\mathbb{C}^*_\hbar$ with the fiber isomorphic to $\mathbf{T}_\Gamma$, roughly, by "correcting" the trivial fiber bundle by means of Stokes automorphisms which depend on the stability data.

This bundle can be extended analytically to $\mathbb{C}_\hbar$ and it has a canonical trivialization at $\hbar = 0$. Then the Taylor series at $\hbar = 0$ of an analytic section of the bundle is resurgent, i.e. it is divergent, but Borel resummable. This is explained in our arXiv:2005.10651. In the case of exponential integrals with Morse function f with different critical values $z_i, 1 \leq i \leq k$, one has $\Gamma = \mathbb{Z}^k$, and $Z(e_i) = z_i, 1 \leq i \leq k$ for the standard basis $e_i, 1 \leq i \leq k$ of $\mathbb{Z}^k$. Then the gluing (Stokes) automorphisms in the standard coordinates $(x_1, ..., x_k)$ on $\mathbf{T}_\Gamma$ have the form $x_i \mapsto x_i(1 + n_{ij}e^{-Z(\gamma_{ij})/\hbar}x^{\gamma_{ij}})$, where $\gamma_{ij} = e_j - e_i, x^{\gamma_{ij}} = x_j x_i^{-1}$. The integer $n_{ij} \in \mathbb{Z}$ which we saw few slides ago can be also interpreted as the intersection index of two opposite thimbles. The Taylor expansions of $I^{mod}(\hbar)$ at $\hbar = 0$ is resurgent.

Generalizations

a) Families of functions, i.e. the function f is replaced by a family $f_u, u \in U$, e.g. $U = \mathbb{C}$, and $f(x_1, ..., x_n, u) = \frac{x_1^3}{3} - ux_1 + \sum_{i \geq 2} x_i^2$. Then we have a wall-crossing structure over $U \times \mathbb{C}_\hbar^*$.
b) Exact 1-form df is replaced by an arbitrary closed 1-form α. There is a generalization of the story with Betti and de Rham cohomology, but it is more interesting and complicated, since the form α can have non-trivial periods. From the point of view of HFT we have two complex Lagrangian submanifolds in T^*X, namely $L_0 = X, L_1 = graph(\alpha)$. Then we can take e.g. $\alpha = (\frac{1}{x} - 1)dx$ as 1-form on $\mathbb{C}^* \subset \mathbb{CP}^1$ and integrate it over the thimble $L = (0, +\infty)$ with the volume form on $\mathbb{C}^*$ given by dx/x. On L we have $\alpha = df, f = log(x) + 1 - x$. Then the corresponding version of the modified exponential integral for $\hbar > 0$ becomes $I^{mod}(\hbar) = \frac{1}{\sqrt{2\pi\hbar}} \int_L e^{\frac{1}{\hbar} f(x)} \frac{dx}{x} = \frac{\Gamma(\lambda)}{\sqrt{2\pi} e^{-\lambda} \lambda^{\lambda - 1/2}}$, where $\lambda = 1/\hbar$. This expression belongs to $\mathbb{C}[[\hbar]]$ and gives rise to a resurgent series.

Infinite-dimensional exponential integrals

Floer complex originally was defined as Morse complex on the infinite-dimensional space of paths connecting two Lagrangian manifolds. In the case when Lagrangian manifolds are holomorphic, we can invert the logic and interpret the path integral in terms of the corresponding DQ-modules. Then we can speak about e.g. WKB-expansions of wave functions or spectra of quantum Hamiltonians and study resurgent properties of arising series by methods of HFT.

Remark

An alternative approach is to study the corresponding infinite-dimensional structures directly. Then one can speculate about the "infinite-dimensional exponential periods" or "variation of Hodge structure of infinite rank", etc. In this way one can construct the wall-crossing structures, but the proof of analyticity will be difficult. E.g. for complexified Chern-Simons one has to count the virtual number of solutions to the Kapustin-Witten equation in $4d$.

Infinite-dimensional space of paths

Consider a complex symplectic manifold $(M, \omega^{2,0})$ and a pair of complex Lagrangian submanifolds L_0, L_1. Assume that $L_0 \cap L_1$ is an analytic subset of M. Let $P(L_0, L_1)$ be the set of real smooth path $\varphi : [0,1] \to M$ such that $\varphi(0) \in L_0, \varphi(1) \in L_1$. This is an infinite-dimensional smooth manifold. We denote by $\omega = Re(\omega^{2,0})$ the real symplectic form on M.The manifold $P(L_0, L_1)$ carries a closed 1-form $\eta = \int_Z \omega^{2,0}(f(t), s(t))$, where $f(t) \in P(L_0, L_1)$, and $s(t)$ is a tangent vector at the point $\varphi(t)$ (which can be identified with a small perturbation of the path $\varphi(t)$). Here Z is a 2-dimensional real cycle bounded by paths $f(t)$ and $s(t)$ and arbitrary real paths in L_0, L_1 connecting their endpoints (since $L_i, i = 0, 1$ are Lagrangian, the integral does not depend on the choice of the latter). Zeros of η are constant paths which are maps of the interval $[0,1]$ to the intersection $L_0 \cap L_1$.

Path integrals and holonomic DQ-modules

In the case of a general complex symplectic manifold $(M, \omega^{2,0})$ the path integral with the action S and boundary conditions on L_0, L_1 can be symbolically written as

$$I = lim_{n \to \infty} \int_{\varphi \in P(L_0, L_1)} e^{S(\varphi)/\hbar} \psi_0(\varphi(t_0))\psi_1(\varphi(t_1))...\psi_n(\varphi(t_n))\mathcal{D}\varphi,$$

where $0 = t_0 < t_1 < t_2 < ... < t_n = 1$ are marked points on the interval $[0,1]$, $\psi_i(t_i)$ are "observables", and $\mathcal{D}\varphi$ is the ill-defined "Feynman measure" on the space of maps $P(L_0, L_1)$.

The axiomatic approach on the next slide gives a meaning to this ill-defined infinite-dimensional integral in terms of finite-dimensional data, associated with M, L_0, L_1. It is inspired by HFT. In this dictionary we interpret ψ_0 and ψ_n as "boundary conditions, i.e. sections of the line bundles $K_{L_0}^{1/2}$ and $K_{L_1}^{1/2}$ respectively, while the "bulk values" $\psi_i(\varphi(t_i)), 0 < i < n$ are interpreted as elements of the quantized algebra $\mathcal{O}_\hbar(M) = \Gamma(M, \mathcal{O}_{\hbar,M})$, where $\mathcal{O}_{\hbar,M}$ is the sheaf of quantized functions. Also we encode a choice of the integration cycle.

For simplicity we consider the case of trivial Hamiltonian. Then the data are:
1) an element $\mu \in Ext^n_{Hol_\hbar(M)}(E_0^{DR,\hbar}, E_1^{DR,\hbar})$ encoding the volume form vol_X, where $n = \frac{dim_{\mathbb{C}} M}{2}$;
2) a class $\gamma \in Ext^0_{\mathcal{F}_{\hbar,loc}}(E_0^{Betti,\hbar,loc}, E_1^{Betti,\hbar,loc})$ encoding the integration cycle.
Then the corresponding exponential integral considered as a formal power series in $\hbar$ (i.e. the perturbative expansion) is given by $I_{form}(\hbar) = \langle RH_\hbar(\mu), \gamma \rangle$, where $\langle \bullet, \bullet \rangle$ is the "Calabi-Yau pairing" between Ext^n and Ext^0 in the n-dimensional Calabi-Yau category $\mathcal{F}_{\hbar,loc}$ and $RH_\hbar$ is the Riemann-Hilbert functor.

In order to obtain from the formal series $I_{form}(\hbar)$ the analytic function $I(\hbar)$ one should apply the Stokes automorphisms, as we did in the case of finite-dimensional exponential integrals. Therefore resurgence of the formal power series is determined by finite-dimensional data (bounds for virtual numbers of pseudo-holomorphic discs). These abstract data have very concrete meaning in examples. E.g. for $M = T^*X$ and $L_0 = X, L_1 = graph(df)$ with f being Morse, we have a basis in $Ext^0_{\mathcal{F}_{\hbar,loc}}(E_0^{Betti,\hbar,loc}, E_1^{Betti,\hbar,loc})$ consisting of local thimbles associated with the critical points of f. Similarly $Ext^n(\mathcal{O}_\hbar(M), E_1^{DR,\hbar}) \simeq H^n_{DR,\hbar}(X)$. In the end we obtain the analytic vector bundle associated with the exponential integral of f.

Recall that we are dealing with Hamiltonian $H = H(\mathbf{q}, \mathbf{p}, t) \equiv 0$, i.e.

$$S(\varphi) = \int_0^1 \sum_i p_i(t) \frac{dq_i(t)}{dt}.$$

We now define in the formal path integral over the local Lefschetz thimble to be equal to the pairing

$$\int e^{\frac{S(\varphi)}{\hbar}} \mathcal{D}\varphi = \langle \psi' | \psi \rangle,$$

where in the RHS we have the pairing of wave functions (i.e. cyclic vectors of the corresponding DQ-modules).

All this can be generalized to the case $H \neq 0$, i.e. for the action

$$S(\varphi) = \int_0^1 \sum_i p_i(t) \frac{dq_i(t)}{dt} + \int_0^1 H(\mathbf{q}(t), \mathbf{p}(t), t) dt.$$

Then we define the LHS as $\langle \psi' | exp(-\widehat{H}) | \psi \rangle$. Axiomatically this can be done via our formalism of wave functions and quantum parallel transport. In particular we can transport the WKB-wave function for L_0 via the flow determined by $H_t = H(\mathbf{q}(t), \mathbf{p}(t), t)$. After that we can use the formalism of analytic stability data for studying resurgence of the series which is the infinite-dimensional version of the formal expansion of the exponential integral

$$\int_{\substack{\text{local Lefschetz thimble} \\ \text{outcoming from } \varphi_\alpha}} e^{\frac{S(\varphi)}{\hbar}} \mathcal{D}\varphi \underset{\hbar \to 0}{\sim} e^{\frac{S(\varphi_\alpha)}{\hbar}} \hbar^{-\frac{n}{2}} \cdot (c_{0,\alpha} + c_{1,\alpha}\hbar + c_{2,\alpha}\hbar^2 + \dots),$$

where $\{\varphi_\alpha\}$ are critical points of $S(\varphi)$.